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Thermodynamics of Noncritical M-Theory and the Topological A-Model

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Abstract

In hep-th/0508024, noncritical M-theory for two-dimensional Type 0A and 0B strings was defined in terms of a double-scaled theory of nonrelativistic fermions in $2+1$ dimensions. Here we study this noncritical M-theory at finite temperature. We derive the exact expression for the free energy of its vacuum solution, as a function of a coupling constant g_M and the radius R of the thermal circle. We show that at high temperature, the theory is effectively described by another M-theory solution, whose effective loop-counting coupling scales in a novel way characteristic of M-theory, as T^3. Our calculations further suggest that noncritical M-theory is dual to the closed string theory of the topological A-model on a Calabi-Yau, with the radius R of the Euclidean time circle in M-theory playing the role of the string coupling constant of the A-model. In this correspondence, T-duality on the Euclidean time circle of noncritical M-theory implies an S-duality for the topological A-model.

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1. Introduction and Summary

Noncritical string theories in two spacetime dimensions (see [2-8] for reviews) provide a useful laboratory in which many features of full string theory can be studied in a simpler, exactly solvable setting. One aspect of the full theory which continues to be shrouded in mystery is its M-theory regime, where fundamental strings are no longer the correct degrees of freedom. In this regime, the dynamics of the theory can be described at low energies by an effective supergravity theory, with M2-branes and M5-branes as solitons. When combined with anticipated dualities to various string vacua, this rather incomplete picture still leads to a wealth of useful information about M-theory. However, a more direct understanding of the M-theory degrees of freedom, in which one would not have to resort to a nonperturbative duality, is clearly desirable.

This problem appears to be difficult in full eleven-dimensional M-theory, but it is precisely the type of question that could be addressed first in the simplified setting of two-dimensional string theory. Type 0A and 0B strings in two dimensions [9,10] seem particularly suitable for this purpose. The spectrum of two-dimensional Type 0A string theory contains stable D0-branes, charged under a $U(1)$ RR symmetry. The D0-brane charge can be naturally interpreted as the Kaluza-Klein momentum along an extra S^1 dimension of space, which we interpret as the extra S^1 of "noncritical M-theory" [1]. It is intriguing that in the Fermi-liquid formulation of the eigenvalues of the matrix model, the extra S^1 simply plays the role of the angular variable on a flat two-dimensional plane populated by the fermions. Noncritical M-theory is thus defined in terms of a large N, double-scaling limit of a system of N nonrelativistic fermions in the inverted harmonic oscillator potential on this "eigenvalue plane." We will describe the fermions in the second-quantized framework, as quanta of a spinless fermionic field $\Psi(t,\lambda_1,\lambda_2)$, whose dynamics is governed in the large N limit by the nonrelativistic action

$$S=N\int dt\,d^2\lambda\left(i\Psi^\dagger\frac{\partial\Psi}{\partial t}-\frac{1}{2}\frac{\partial\Psi^\dagger}{\partial\lambda_1}\frac{\partial\Psi}{\partial\lambda_1}-\frac{1}{2}\frac{\partial\Psi^\dagger}{\partial\lambda_2}\frac{\partial\Psi}{\partial\lambda_2}+\frac{1}{2}\omega_0^2(\lambda_1^2+\lambda_2^2)\,\Psi^\dagger\Psi\right). \tag{1.1}$$

Here (λ_1,λ_2) are Cartesian coordinates on the flat eigenvalue plane $\mathbf{R}^2$, and ω_0 is a fundamental frequency of the theory. We will set $\omega_0=1$ throughout the paper.

In noncritical M-theory so defined, one can find two-dimensional Type 0A and 0B string vacua as solutions (as we will review briefly in Section 2.1). In addition, the space of all solutions also includes a "true ground state" of the system, in which the lowest N single-particle states have been filled by the N available fermions, up to some distance μ from the top of the potential. In the double-scaling limit, which is the semiclassical limit of the Fermi liquid described by (1.1), N is sent to infinity while the Fermi energy μ is held fixed. $g_M=1/\mu$ plays the role of a coupling constant in this vacuum. Some properties of this vacuum of noncritical M-theory were studied in [1].

In this paper, we intend to probe this solution further, with the hope of learning more about the nature of its collective excitations. For any given physical system, useful information about its underlying degrees of freedom can be revealed by exposing the system to extreme physical conditions, such as high temperatures, high-energy scattering, or strong fields. This proven strategy was used, for example, in the early days of superstring theory: A series of gedanken experiments was designed (see, *e.g.*, [11-15]) to reveal the underlying degrees of freedom of the theory, or its hypothetical "unbroken phase" in which large underlying symmetries could become manifest. In this paper, we shall apply this strategy to the vacuum state of noncritical M-theory, and examine it at finite temperature.[1]

In quantum field theory, it is well-known that the high-temperature behavior is formally governed by a field theory in one fewer spacetime dimension, with the effective loop-counting coupling that scales as T. Frequently, this naive picture is strongly modified by renormalization effects. In string theory at finite temperature [17-21,14,15], on the other hand, the high-temperature behavior of one string vacuum is formally mapped by T-duality to the low-temperature behavior of another string vacuum in the same spacetime dimension. The effective loop-counting coupling of this dual string theory scales as T^2 [14]. In most string vacua, this naive picture is modified by the existence of the Hagedorn phase transition, where the canonical ensemble description breaks down, and one needs to resort to the microcanonical ensemble.

One naturally wonders what is the high-temperature behavior of M-theory; in particular, is it more akin to field theory, or string theory, or neither? In eleven dimensions, this question again seems very difficult to answer, since we lack the required control over the nonsupersymmetric compactifications of M-theory that would formally define its thermodynamic ensemble in the regime of interest.[2] On the other hand, the same question can be addressed in noncritical M-theory in $2 + 1$ dimensions. In Section 2, the exact partition function of the vacuum solution of noncritical M-theory at finite temperature will be evaluated, and its high temperature behavior studied. We shall find the following:

- The behavior of the M-theory vacuum state at high temperatures is governed by another, dual solution of noncritical M-theory, related to the original vacuum by a symmetry similar to T-duality, which noncritical M-theory inherits from the underlying T-duality between Type 0A and 0B strings on a compact Euclidean time circle.
- In the high-energy limit, the free energy of the system scales as T^3, *i.e.*, in a way characteristic of a massless relativistic field theory in $2 + 1$ dimensions.
- In this dual M-theory solution governing the high-temperature regime, the effective loop-counting coupling scales at high temperature as T^3. This is a novel scaling

[1] Aspects of the high-energy behavior of noncritical M-theory will be studied elsewhere [16].

[2] Some papers discussing critical M-theory at finite temperature include [22,23]; see also [24].

behavior, unlike in field theory ($\sim T$) or string theory ($\sim T^2$).

Our results in Section 2 will be obtained using the grand-canonical partition function of noncritical M-theory at finite temperature T. The grand-canonical ensemble turns out to be equivalent, albeit in a somewhat subtle way involving intricacies of the double-scaling limit, to the canonical ensemble. Once this equivalence is established, one can ignore the thermodynamic interpretation of the partition function, and simply interpret it as the partition function of noncritical M-theory in Euclidean signature, with the Euclidean time dimension compactified on a circle of radius $R = 1/(2\pi T)$.

This Euclidean compactification of noncritical M-theory now has *two* $U(1)$ isometries: the rotation on the spatial eigenvalue plane, and the Euclidean time translation. The angular dimension on the eigenvalue plane has been interpreted as the "M-theory dimension" for two-dimensional Type 0A strings, with the angular momentum on the plane playing the role of the D0-brane charge in Type 0A string theory. In the next step, it is natural to ask whether the Euclidean time dimension can also be interpreted as the "M-theory dimension" of some string theory. This question will be analyzed in Section 3. Somewhat surprisingly, we find evidence that this question has a simple answer: *The string theory associated with the reduction of noncritical M-theory along the Euclidean time circle is the closed string theory of the topological A-model on the resolved conifold!*

The thermal ensemble of noncritical M-theory naturally corresponds to the path integral with the antiperiodic condition on the Fermi field Ψ around the Euclidean time t_E. In the study of the relation to the A-model, it is more natural to relax this condition, and study noncritical M-theory with Ψ periodic up to an arbitrary phase b,

$$\Psi(t_E + 2\pi R, \lambda_1, \lambda_2) = e^{ib}\Psi(t_E, \lambda_1, \lambda_2). \tag{1.2}$$

We find that this entire class of Euclidean compactifications of noncritical M-theory is related to the topological A-model. In this correspondence, the radius R of the Euclidean time circle, the coupling constant g_M and the phase b of noncritical M-theory are related to the A-model string coupling g_A and the Kähler modulus t_A of the resolved conifold by a simple duality relation,

$$g_A = 2\pi i R, \qquad t_A = \frac{2\pi R}{g_M} + ib. \tag{1.3}$$

Note that what is a coupling constant on one side of the duality becomes a geometric parameter on the other side of the duality, and vice versa.

This duality between noncritical M-theory and the topological A-model can be expected to have a wide range of interesting implications. Some of them are:

- The relation between the string coupling g_A and the radius R of the M-theory circle suggests that noncritical M-theory can play the role of the topological M-theory [25,26] for the topological strings of the A-model.

- Noncritical M-theory provides a nonperturbative completion of the topological closed string theory of the A-model.
- T-duality of noncritical M-theory (inherited from the underlying T-duality of two-dimensional Type 0A and 0B strings [27]) implies an S-duality for the topological A-model.
- In combination with the Gopakumar-Vafa duality [28,29], our duality relates noncritical M-theory to Chern-Simons gauge theory on S^3.
- In the melting crystal interpretation of the quantum foam phase of Calabi-Yau [30], the temperature of the crystal turns out to be equal to the temperature of the noncritical M-theory vacuum. The constituents of the crystal and the quantum foam of the Calabi-Yau appear intimately related to the constituent fermions of noncritical M-theory.

2. Noncritical M-Theory at Finite Temperature

Noncritical M-theory for two-dimensional Type 0A and 0B strings is defined by the nonrelativistic Lagrangian (1.1) only formally. The proper definition, discussed in detail in [1], begins with a finite number N of fermions, and with the inverted harmonic oscillator potential replaced with a more general regulating potential $V(\lambda_1, \lambda_2)$. This potential includes stabilizing anharmonic terms which ensure that the spectrum is bounded from below. The simplest way of mimicking such terms is to place an infinite wall at some distance from the origin, which cuts off the single-particle energy spectrum from below at some $-\Lambda$; this will be the regulator assumed throughout this paper. In the large N limit, the anharmonic terms are scaled away (or Λ is taken to infinity), and the only relevant piece of the potential in this double-scaling limit is its inverted harmonic oscillator part.

2.1. Noncritical M-theory for Type 0A and 0B Strings in Two Dimensions

Both Type 0A and 0B string vacua in two dimensions are solutions of this theory. In order to see that, note that the Hamiltonian associated with (1.1) can be viewed from two complementary perspectives.

In the polar coordinates (λ, θ) on the eigenvalue plane, the Fermi field Ψ (and its conjugate momentum $\Psi^\dagger$) can be decomposed into eigenfunctions of the single-particle angular momentum operator J, $\Psi(\lambda,\theta) = \sum_{q\in\mathbf{Z}} e^{iq\theta}\Psi_q(\lambda)$. Consequently, the second-quantized Hamiltonian decomposes into an infinite sum,

$$\mathcal{H}(\Psi,\Psi^\dagger) = \sum_{q\in\mathbf{Z}} \mathcal{H}_q(\Psi_q,\Psi_q^\dagger). \tag{2.1}$$

Each individual term $\mathcal{H}_q$ is equivalent to the Hamiltonian of the Type 0A string theory in the linear dilaton vacuum with RR flux q.

In the Cartesian coordinates (λ_1, λ_2), the single-particle Hamiltonian $h = h_1 + h_2$ describes two decoupled oscillators. The energy h_2 of (say) the second oscillator is a conserved quantity. We can decompose Ψ into a complete basis of eigenfunctions $\psi_\nu(\lambda_2)$ of h_2. Since the eigenvalues ν of h_2 are continuous, we obtain an integral[3]

$$\Psi(\lambda_1, \lambda_2) = \int d\nu\, \Psi_\nu(\lambda_1)\, \psi_\nu(\lambda_2). \tag{2.2}$$

As a result, the second-quantized Hamiltonian $\mathcal{H}$ decomposes into an integral over ν of a one-parameter family of decoupled Hamiltonians $\mathcal{H}_\nu$, each of which depends only on a single second-quantized field $\Psi_\nu(\lambda_1)$. Each member of this family of Hamiltonians is essentially equivalent to the Hamiltonian of two-dimensional Type 0B string theory in Fermi liquid representation.

Having understood these facts about noncritical M-theory, it is now clear how the Type 0A and 0B vacua can be constructed as exact solutions of this theory. The Type 0A linear dilaton vacuum with RR flux q is given simply by the state of the $2+1$ dimensional noncritical M-theory in which all single-particle states with angular momentum $J = q$ are occupied up to some (double scaled) fermi energy $\mu = -N\epsilon_F$, while all fermion states with $J \neq q$ are empty. The physics of this solution is *exactly equivalent* to the physics of the linear dilaton ground state of the Type 0A theory in the Fermi liquid representation. In particular, since the Fermi sea of this state is empty in all sectors with $J \neq q$, all excitations in those sectors are infinitely energetic from the perspective of the Type 0A ground state, and they decouple in the double-scaling limit. The physics of the remaining excitations is then isomorphic to that of Type 0A string theory in the Fermi liquid picture.

Similarly, the Type 0B linear dilaton vacuum corresponds to the Fermi sea in which the N fermions occupy (up to the Fermi energy μ) only the single-particle states whose eigenvalue of h_2 is equal to some fixed ν, while all single-particle states whose eigenvalue $h_2 \neq \nu$ are empty. In the double-scaling limit, all excitations with $h_2 \neq \nu$ are infinitely energetic and decouple. The physics of the remaining finite-energy excitations of this solution is precisely equivalent to the Fermi liquid representation of the Type 0B string theory. In the Type 0A and 0B string vacua of noncritical M-theory, the fundamental frequency ω_0 plays the role of $1/\sqrt{2\alpha'}$ [1].

In addition to reproducing such known two-dimensional string vacua, noncritical M-theory has other interesting solutions. The one of primary interest to us is the "true"

[3] More precisely, for each eigenvalue of h_2 there are two eigenstates, one for each parity $\pm$ under $\lambda_2 \to -\lambda_2$. We implicitly include the sum over parity in our definition of the integral over ν.

ground state of the theory, in which the N available fermions simply occupy *the lowest N single-particle states* of the theory, irrespective of what other quantum numbers they may carry (and with the Fermi energy μ held fixed in the large N limit).

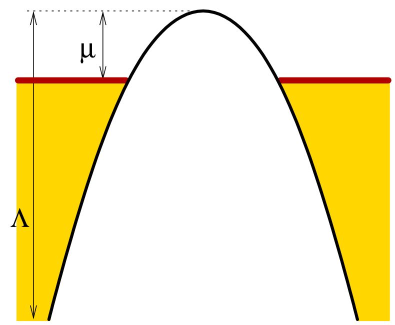

Fig. 1: The ground state of noncritical M-theory in the Fermi liquid representation. The inverted harmonic oscillator potential is rotationally invariant on the eigenvalue plane, with the second dimension suppressed in the figure. All single-particle states are occupied from the cutoff up to the top of the Fermi sea, some distance μ below the top of the potential.

In this simplified context of noncritical M-theory, it is this natural vacuum state which is perhaps the closest analog of the eleven-dimensional vacuum of full M-theory. Various properties of this vacuum state of noncritical M-theory were studied in [1], and we will review some of them in Section 2.5 below.

It is also worthwhile to point out that families of static solutions interpolating between the M-theory vacuum and the Type 0A or 0B string theory can be constructed. Starting from Type 0A (or 0B) solutions as described above, one can raise the Fermi surface in sectors with $J \neq q$ (or $h_2 \neq \nu$) to some finite distance below the top of the potential, making the excitations in those sectors finitely energetic. This change in the Fermi surface represents a hidden deformation parameter of two-dimensional Type 0A and 0B string theory.

2.2. The Grand Canonical Ensemble

We find it convenient to analyze the thermodynamics of noncritical M-theory using the grand canonical ensemble of the fermions. In this ensemble, the central object of our interest will be the thermodynamic potential $\Gamma(\beta, \mu)$, which is a function of the inverse temperature $\beta = 1/T$ and the (double-scaled) chemical potential μ associated with the conserved number of fermions N. $\Gamma(\beta, \mu)$ is defined in terms of the grand canonical partition function $\mathcal{Z}_M(\beta, \mu)$ of the noncritical M-theory vacuum,

$$\Gamma(\beta, \mu) = \frac{1}{\beta} \log \mathcal{Z}_M = \frac{1}{\beta} \int_{-\infty}^{\infty} d\xi \, \rho(\xi) \log \left(1 + e^{\beta(\mu - \xi)} \right) . \tag{2.3}$$

Here $\log \mathcal{Z}_M$ has been related to the single-particle density of states $\rho(\xi)$ at zero temperature and at energy ξ, a quantity studied in detail in [1].

In the thermodynamic limit, which is a part of the double-scaling limit, this ensemble is equivalent to the canonical ensemble provided the fluctuations in the number of particles are suppressed, a condition that we will return to in Section 2.4 below. Potentially, the canonical ensemble in string theory is known to suffer from several instabilities, including the Jeans instability common to all gravitating systems, and the stringy instability due to the Hagedorn phase transition. These destabilize the canonical ensemble and require the full power of the microcanonical ensemble. Such instabilities do not occur in Type 0A and 0B string theories in two dimensions, and we will see that they do not occur in noncritical M-theory either. The microcanonical, canonical and grand canonical ensembles are all equivalent, but the argument is surprisingly subtle and involves the double-scaling limit in a nontrivial way.

In [1], an exact formula for the density of states $\rho(\xi)$ in the M-theory vacuum was derived,

$$\begin{aligned}\rho(\xi) &= \frac{1}{4\pi}\,\mathrm{Re}\int_0^\infty d\tau e^{-i\xi\tau}\,\frac{1}{\sinh^2(\tau/2)}\\ &= -\frac{\xi}{2\tanh(\pi\xi)}+\frac{\Lambda}{2}.\end{aligned}\tag{2.4}$$

The integral representation for the density of states in (2.4) is divergent, but the divergence is rather mild, with the divergent piece independent of μ and resulting in a simple dependence of $\rho(\xi)$ on the cutoff Λ. In the double scaling limit, Λ is scaled to infinity with N. As in [1], we only keep the leading dependence on Λ, dropping all terms $\mathcal{O}(1/\Lambda)$.

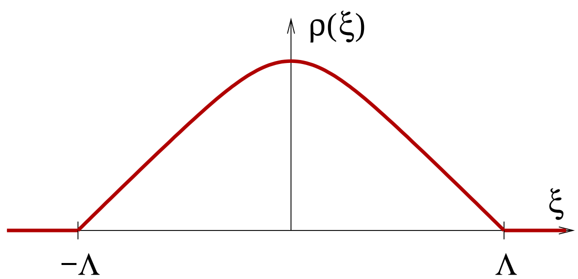

Fig. 2: The density of states (2.4) as a function of the single-particle energy ξ of the fermions. The density is invariant under $\xi \to -\xi$. Notice also the importance of the nonuniversal ξ-independent constant in (2.4): if we ignored this Λ-dependent term, the density of states would be negative.

Similarly, the formal expression (2.3) for Γ is divergent. In most of our arguments, we will concentate only on the universal part of all the thermodynamic variables. Thus, in order to eliminate all dependence on Λ and isolate the universal information in Γ, we

use a trick standard in noncritical string theory, and evaluate

$$\frac{\partial^3\Gamma}{\partial\mu^3} = -\beta^2 \int_{-\infty}^{\infty} d\xi \, \frac{\partial\rho(\xi)}{\partial\xi} \, \frac{1}{4\cosh^2\left[\beta(\xi-\mu)/2\right]}, \tag{2.5}$$

which is cutoff-independent. We will return to the dependence of the thermodynamic quantities on the cutoff in the next subsection.

Using the integral representation (2.4) for $\rho(\xi)$ and performing the integral over ξ by residues, we get

$$\begin{aligned}\frac{\partial^3\Gamma}{\partial\mu^3} &= -\frac{1}{4\pi}\int_{-\infty}^{\infty} d\xi \ \mathrm{Re}\int_0^{\infty} d\tau\, e^{i\xi\tau}\frac{\tau^2}{\sinh^2(\tau/2)}\,\frac{1}{1+e^{2\pi R(\xi-\mu)}}\\ &= -\frac{1}{2\pi}\ \mathrm{Im}\int_0^{\infty} d\tau\, e^{i\mu\tau}\frac{\tau/2}{\sinh^2(\tau/2)}\,\frac{\tau/(2R)}{\sinh\left[\tau/(2R)\right]}.\end{aligned} \tag{2.6}$$

This integral representation for the thermodynamic potential of the M-theory vacuum is similar to the corresponding expressions in noncritical string theory [31] (see also [2,4] for recent reviews). Note that – unlike for example in noncritical bosonic string theory – the M-theory formula does not exhibit any manifest form of self T-duality. However, as we will see later, M-theory does inherit a form of T-duality from the underlying T-duality relating Type 0A and 0B strings with RR flux.

2.3. Terms of Low Order in μ

The integral formula (2.6) is convergent, but it determines $\Gamma(\beta,\mu)$ only up to a second order polynomial in μ,

$$\Gamma_0(\beta) + \Gamma_1(\beta)\,\mu + \Gamma_2(\beta)\,\frac{\mu^2}{2}. \tag{2.7}$$

The coefficients $\Gamma_i(\beta)$ are cutoff-dependent, and divergent in the double-scaling limit. However, before we dismiss them as non-universal, we should note that they do contain some valuable physical information. Imagine, for example, that one is interested in the behavior of our system at finite temperature, and for zero chemical potential $\mu = 0$. In such circumstances, the thermodynamic potential Γ would be given entirely by the lowest term $\Gamma_0(\beta)$ in (2.7) (at least if we assume analyticity at $\mu = 0$).

With such applications in mind, it is useful to separate the universal information in $\Gamma_i(\beta)$ from the nonuniversal terms. Even though $\Gamma_i(\beta)$ are all divergent as we take the cutoff to infinity, we can take additional derivatives with respect to β, and obtain

$$\frac{\partial^2(\beta\Gamma_0)}{\partial\beta^2} = \frac{1}{4}\int_{-\infty}^{\infty} d\xi\,\rho(\xi)\frac{\xi^2}{\cosh^2(\beta\xi/2)} = \frac{\Lambda}{2\beta} - \frac{1}{8}\int_{-\infty}^{\infty} d\xi\,\frac{\xi^3}{\tanh(\pi\xi)\,\cosh^2(\beta\xi/2)}. \tag{2.8}$$

In the last step, we have used the expression (2.4) for $\rho(\xi)$, and evaluated the convergent integral that multiplies the Λ term. The remaining integral is now also convergent, and

(2.8) determines the dependence of $\beta\Gamma_0(\beta)$ on β up to a (possibly divergent) nonuniversal polynomial linear in β.

Similarly,

$$\frac{\partial\Gamma_1}{\partial\beta} = -\frac{1}{4}\int_{-\infty}^{\infty} d\xi\,\rho(\xi)\frac{\xi}{\cosh^2(\beta\xi/2)} = 0. \tag{2.9}$$

This integral vanishes because $\rho(\xi)$ is an even function of ξ, implying that Γ_1 is independent of β, and equal to a (possibly divergent) nonuniversal constant.

As to the Γ_2 term, its divergence comes completely from the linear divergence of ρ, and we can evaluate it without taking any additional derivatives,

$$\Gamma_2 = \beta\int_{-\infty}^{\infty} d\xi\,\rho(\xi)\frac{1}{\cosh^2(\beta\xi/2)} = \frac{2\pi^2\Lambda}{3\beta^2} - \frac{\beta}{2}\int_{-\infty}^{\infty} d\xi\,\frac{\xi}{\tanh(\pi\xi)\,\cosh^2(\beta\xi/2)}. \tag{2.10}$$

Despite being non-universal, the leading Λ-dependent terms in (2.8) and (2.10) will play an important role in our discussion of the equivalence of various thermodynamic ensembles in the next subsection.

2.4. Remarks on Thermodynamic Stability

In order to substantiate the discussion of the grand canonical ensemble, we must examine whether this ensemble is well-defined, and if so, whether it is equivalent to the canonical and the microcanonical ensemble. In order to test that, we will test the validity of the fluctuation-dissipation theorem. This theorem relates the fluctuations in the mean energy $\langle E\rangle$ and the mean number of particles $\langle N\rangle$ to thermodynamic quatities: the specific heat and the isothermal compressibility. In terms of Γ, we have

$$\langle N^2\rangle - \langle N\rangle^2 = \frac{1}{\beta}\frac{\partial^2\Gamma}{\partial\mu^2}. \tag{2.11}$$

Similarly, the specific heat C_V is the measure of the energy fluctuations,

$$\langle E^2\rangle - \langle E\rangle^2 = \frac{C_V}{\beta^2} \equiv \frac{\partial^2(\beta\Gamma)}{\partial\beta^2}. \tag{2.12}$$

First of all, in order for the grand canonical ensemble to be well-defined, both of these quantities need to be positive. Using (2.11) and (2.12) together with our evaluation of the leading divergent terms in (2.8) and (2.10), we deduce that

$$\langle N^2\rangle - \langle N\rangle^2 = \frac{\Lambda}{2\beta} - \frac{1}{8}\int_{-\infty}^{\infty} d\xi\,\frac{\xi}{\tanh(\pi\xi)\,\cosh^2[\beta(\xi-\mu)/2]}, \tag{2.13}$$

and

$$\langle E^2\rangle - \langle E\rangle^2 = \frac{\Lambda}{2\beta}\left(\frac{\pi^2}{3\beta^2} + \mu^2\right) - \frac{1}{8}\int_{-\infty}^{\infty} d\xi\,\frac{\xi^3}{\tanh(\pi\xi)\,\cosh^2[\beta(\xi-\mu)/2]}. \tag{2.14}$$

As we see, if we discarded the nonuniversal Λ-dependent pieces in (2.13) and (2.14) and kept only the universal terms dependent on μ, both quantities would be negative, suggesting a possible thermodynamic instability of the canonical as well as the grand canonical ensemble. The proper behavior of the ensembles is restored by the leading nonuniversal term proportional to the cutoff Λ in (2.13) and (2.14). These terms are always positive, and stabilize both ensembles in the double-scaling limit.

Having established the stability of the ensembles, it is also straightforward to verify that the fluctuations of the mean values $\langle N\rangle$ and $\langle E\rangle$ are small in the thermodynamic limit, *i.e.*, that

$$\frac{\langle N^2\rangle - \langle N\rangle^2}{\langle N\rangle^2} \quad \text{and} \quad \frac{\langle E^2\rangle - \langle E\rangle^2}{\langle E\rangle^2} \tag{2.15}$$

vanish. In the thermodynamic limit, we take $\Lambda \to \infty$ while keeping μ and β fixed, the denominators in (2.15) indeed diverge faster than (2.13) and (2.14), and all three ensembles are equivalent. (In view of this, we can now drop the brackets in $\langle N\rangle$ and $\langle E\rangle$, and denote them simply by N and E.)

2.5. Expansion in the Powers of the Coupling $1/\mu$

In the matrix models of two-dimensional string theory, the inverse Fermi energy $1/\mu$ plays the role of the string coupling, $g_s = 1/\mu$. The exact partition function can be expanded as an asymptotic series[4] in the powers of g_s,

$$F_{\text{string}} \approx \sum_{h=0}^{\infty} F_h(R)\, g_s^{2h-2} \tag{2.16}$$

and compared to the perturbative string-theory calculation.

In the vacuum of noncritical M-theory, similarly, the inverse Fermi energy plays the role of an "M-theory coupling constant," which we denote by[5]

$$g_M = 1/\mu. \tag{2.17}$$

One can expand physical quantities in the powers of g_M, and contrast the behavior of this expansion with the expansion (2.16) in two-dimensional string theory. In (2.16), the higher

[4] Throughout this paper, we use the symbol "$\approx$" to denote exact asymptotic expansions of various formulas.

[5] Folklore says that M-theory in eleven dimensions contains no dimensionless couplings. However, on a nontrivial background geometry with a characteristic length scale $\mathcal{R}$, a natural dimensionless coupling emerges as $\mathcal{R}$ measured in Planck units. We believe that g_M should be thought of as a dimensionless coupling of such a geometric origin. In fact, a geometric interpretation for g_M will indeed be found in Section 3.

genus coefficients F_h, with $h > 1$, are universal functions of the geometric moduli, such as the radius R of the Euclidean time circle. In addition, the lowest genus terms F_0 and F_1 also depend logarithmically on μ and on the cutoff Λ. Moreover, if open strings are present, odd powers of g_s can also appear in (2.16). In this sense, the string perturbation expansion is naturally an expansion in "halves of loops." More precisely, we wish to interpret (2.16) as a semiclassical expansion of an effective action $S_{\rm eff}/\hbar$ in powers of $\hbar$. The power of $\hbar$ counts by definition the number of loops. The leading-order term in (2.16) is by definition classical, and therefore of order $1/\hbar$. The one-loop term in $S_{\rm eff}/\hbar$ is independent of $\hbar$. Thus, we see the well-known fact that the role of $\hbar$ (or the "loop-counting parameter") in string theory is effectively played by g_s^2. Since the disk diagram would be proportional to $1/g_s$, this diagram effectively contributes at "one-half-loop" order compared to the classical term originating from the sphere $1/g_s^2$.

In the case of the vacuum energy of the M-theory vacuum state at zero temperature, this expansion was studied in detail in [1], and found to take the form

$$\mathcal{F}_M \approx \sum_{n=0}^{\infty} \mathcal{F}_n \, g_M^{n-3}. \tag{2.18}$$

This expansion exhibits several intriguing features [1]:

- The leading term in (2.18) is of order $1/g_M^3$. In the semiclassical expansion, this will be the leading, classical term, implying that the role of the Planck constant in noncritical M-theory is played by $\hbar \sim g_M^3$. Since the subleading corrections found in [1] are integer powers of g_M, the natural expansion is in "thirds of loops," or $\hbar^{1/3} \sim g_M$. This is intriguingly reminiscent of heterotic M-theory [32] in eleven dimensions.
- Unlike in two-dimensional Type 0A/0B string theory, where all orders in (2.16) are nontrivial, it turns out that the perturbative expansion (2.18) in noncritical M-theory terminates at one-loop. All coefficients $\mathcal{F}_n$ with $n > 3$, *i.e.*, all terms in (2.18) proportional to positive powers of g_M, are identically zero!
- The instanton corrections to (2.18) were also determined in [1]. They have characteristic nonperturbative weights

  $$\exp\left(-\frac{2\pi n}{g_M}\right) \sim \exp\left(-\frac{2\pi n}{\hbar^{1/3}}\right), \tag{2.19}$$

  with n any positive integer. According to a classic argument [33], nonperturbative effects in string theory are stronger than in field theory, since they scale as $\exp(-C/\hbar^{1/2})$ compared to the field-theory behavior $\exp(-C/\hbar)$. By the same argument, (2.19) indicates that nonperturbative effects in noncritical M-theory are even stronger than in string theory.

- For each n, the instanton term (2.19) is multiplied by a perturbative expression similar to (2.18). This expression starts at one-third loop order $\hbar^{1/3}$ (*i.e.*, the leading, classical term is absent), and is also one-loop exact.

It is easy to show that the one-loop exactness of the asymptotic expansion (2.18) persists at finite temperature as well. The expression (2.6) for the (third derivative of) the thermodynamic potential can be expanded in the powers of $1/\mu$ at fixed R (after changing the integration variable to $\sigma = \mu\tau$ as in [1]). This yields

$$\frac{\partial^3\Gamma}{\partial\mu^3} \approx -\frac{1}{\pi}\int_0^\infty \frac{d\sigma}{\sigma}\sin(\sigma)\left(1+\sum_{n=1}^\infty f_n(R)\frac{\sigma^{2n}}{\mu^{2n}}\right)e^{-\epsilon\sigma}. \tag{2.20}$$

We do not even need to determine the precise form of the R-dependent coefficients $f_n(R)$, because the integrals that they multiply all vanish identically, with the exception of the leading constant term. Thus, the $1/\mu$ expansion (2.20) terminates at the lowest order,

$$\frac{\partial^3\Gamma}{\partial\mu^3} \approx -\frac{1}{2} + \text{nonperturbative terms}, \tag{2.21}$$

which in turn implies that

$$\Gamma(\mu,R) \approx -\frac{1}{12}\mu^3 + \mathcal{F}_1(R)\,\mu^2 + \mathcal{F}_2(R)\,\mu + \mathcal{F}_3(R) + \text{nonperturbative terms}. \tag{2.22}$$

In addition to their dependence on the temperature, the subleading terms $\mathcal{F}_i$, $i = 1,2,3$ may also depend on the nonuniversal cutoff Λ.

We have not checked explicitly the behavior of the perturbative expansions near the instanton corrections, but we expect them to continue to be one-loop exact at finite temperature as well. For the special case of $R = 1$, this will be explicitly verified in Section 2.9.

2.6. The Low-Temperature Expansion

We shall now analyze the thermodynamic behavior of noncritical M-theory in the regimes of low and high temperature.

First, we expand Γ as an asymptotic expansion at low temperature $T = 1/(2\pi R)$ at fixed μ:

$$\begin{aligned}\frac{\partial^3\Gamma}{\partial\mu^3} &\approx -\frac{1}{4\pi}\int_0^\infty d\tau\,\sin(\mu\tau)\,\frac{\tau}{\sinh^2(\tau/2)}\left(1-\sum_{k=1}^\infty \frac{2(2^{2k-1}-1)B_{2k}}{(2k)!}\left(\frac{\tau}{2R}\right)^{2k}\right)\\ &= -\frac{1}{4\pi}\sum_{k=0}^\infty \frac{1}{R^{2k}}\,\frac{(1-2^{1-2k})(-1)^{k-1}B_{2k}}{(2k)!}\,\frac{\partial^{2k}}{\partial\mu^{2k}}\int_0^\infty d\tau\,\frac{\tau}{\sinh^2(\tau/2)}\,\sin(\mu\tau).\end{aligned} \tag{2.23}$$

B_m are the Bernoulli numbers (see [1] for our conventions). The remaining integral in (2.23) can be evaluated and the series formally resummed into the following compact expression,

$$\frac{\partial^3\Gamma}{\partial\mu^3} \approx -\frac{R\left(\frac{1}{2R}\frac{\partial}{\partial\mu}\right)^2}{\sin\left(\frac{1}{2R}\frac{\partial}{\partial\mu}\right)}\frac{\mu}{\tanh(\pi\mu)}, \tag{2.24}$$

which should be interpreted in the sense of an asymptotic expansion in $1/R$. Integrating (2.23) once, we obtain

$$\frac{\partial^2\Gamma}{\partial\mu^2} \approx -\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{R^{2k}}\frac{(1-2^{1-2k})(-1)^{k-1}B_{2k}}{(2k)!}\frac{\partial^{2k}}{\partial\mu^{2k}}\frac{\mu}{\tanh(\pi\mu)}+\frac{\Lambda}{2}. \tag{2.25}$$

This result is in accord with the following general observation. The number of particles N is related to Γ via

$$N=\frac{\partial\Gamma}{\partial\mu}.$$

Taking a derivative of both sides, we obtain

$$\rho_{\text{eff}}(\mu,\beta)\equiv\frac{\partial N}{\partial\mu}=\frac{\partial^2\Gamma}{\partial\mu^2},$$

where $\rho_{\text{eff}}(\mu,\beta)$ is the effective density of states at finite temperature. It is reassuring to see that the leading term in (2.25) is indeed the zero-temperature density of states $\rho(\mu)$, and reproduces the zero-temperature result of [1].

Perhaps the most interesting feature of the low-temperature expansion (2.25) is the fact that all dependence on T in (2.25) falls off rapidly as $\mu>1$. This suggests a radical reduction of the effective number of degrees of freedom in the theory at small values of the M-theory coupling g_M. This is compatible with the one-loop exactness of the weak-coupling expansion observed in [1] and in the previous subsection. Both of these features suggest that the theory becomes effectively topological for small values of g_M, with a smooth crossover to a more dynamical regime at strong coupling.

2.7. The High-Temperature Expansion

At high temperature, we can expand the thermodynamic potential Γ in the powers of R,

$$\frac{\partial^3\Gamma}{\partial\mu^3}\approx\frac{R}{2\pi}\int_0^\infty\frac{d\sigma\,\sigma}{\sinh(\sigma/2)}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}\sum_{k=0}^{\infty}\frac{(2k-1)B_{2k}}{(2k)!}\mu^{2m+1}(\sigma R)^{2m+2k}. \tag{2.26}$$

First, we wish to isolate the behavior of the system as $T\to\infty$. At high temperature, the system can be expected to become effectively classical, and we must find the correct way of

identifying this classical limit. Holding μ while taking the high-temperature limit $R \to 0$ in (2.26) would lead to

$$\frac{\partial^3 \Gamma}{\partial \mu^3} \approx -\frac{\pi \mu R}{2} + \mathcal{O}(R^3). \tag{2.27}$$

Hence, in this naive limit of infinite temperature with μ fixed, $\partial^3\Gamma/\partial\mu^3$ would vanish. In order to obtain a sensible classical limit at high temperatures, we must instead scale μ with R such that

$$\widetilde{\mu} \equiv R\,\mu \tag{2.28}$$

is held fixed as $R \to 0$. This leads to a nontrivial high-temperature behavior,

$$\begin{aligned}\frac{\partial^3 \Gamma}{\partial \mu^3} &\approx -\frac{1}{2\pi}\int_0^\infty d\sigma\, \sin(\widetilde{\mu}\sigma)\left(\frac{1}{\sigma} + \mathcal{O}(R^2)\right)\frac{\sigma}{\sinh(\sigma/2)} \\ &= -\frac{1}{2}\tanh(\pi\widetilde{\mu}) + \mathcal{O}(R^2),\end{aligned} \tag{2.29}$$

in which all orders in $\widetilde{\mu}$ contribute at the leading order in T. Thus, the proper classical limit of noncritical M-theory at high temperature involves holding $\widetilde{\mu}$ fixed as T goes to infinity.

We are now ready to compare the high-temperature behavior of noncritical M-theory to the high-temperature behavior in field and string theory.[6]

In quantum field theory, consider for illustration the case of Yang-Mills gauge theory. The high-temperature limit is formally governed by an effective field theory in one fewer spacetime dimension, with effective loop-counting coupling given by $g^2_{\mathrm{YM,\ eff}} = g^2_{\mathrm{YM}}T$. In order to obtain the proper classical limit at high temperatures, we must keep $g^2_{\mathrm{YM,\ eff}}$ fixed while taking T to infinity. Of course, in most field theories, this naive picture will be strongly modified by renormalization effects and infrared divergences.

String theory at high temperature does not undergo an effective dimensional reduction, and is instead formally governed by a T-dual solution in the same dimension at the dual temperature. The effective loop-counting coupling of the dual theory is given in terms of the original string coupling by $g_s^2T^2$. In order to obtain a consistent classical limit at high temperatures, $g_s^2T^2$ must be held fixed as T goes to infinity. In this limit, as was pointed out in [14], the free energy of the system scales as T^2, *i.e.*, as in a conformal field theory in $1+1$ spacetime dimensions. In critical string theory, this formal picture is strongly modified by the Hagedorn phase transition.

With this picture in mind, it is natural to ask the following questions in noncritical M-theory:

[6] A nice discussion of the high-temperature classical limit in string theory and field theory can be found, for example, in the classic paper by Atick and Witten [14].

(1) What is the effective theory that governs the high-temperature behavior of the noncritical M-theory vacuum?
(2) What is the leading behavior of the thermodynamic potential (or the free energy) at high temperature?
(3) In the effective theory governing the high-temperature behavior, how does the effective coupling scale with T?

We start answering these questions by isolating the leading high-temperature behavior of the thermodynamic potential Γ. As we have seen, $\widetilde{\mu}$ is held fixed in the high-temperature limit. Upon rewriting (2.29) in terms of $\widetilde{\mu}$ and T, we obtain

$$\frac{\partial^3\Gamma}{\partial\widetilde{\mu}^3} = (2\pi T)^3\left(-\frac{1}{2}\tanh(\pi\widetilde{\mu})\right) + \ldots, \tag{2.30}$$

where the "..." denote all terms subleading in T. Thus, at high temperatures, the most divergent term in the thermodynamic potential (and, consequently, also in the free energy) scales as

$$\Gamma(T,\widetilde{\mu}) \sim T^3. \tag{2.31}$$

This high-temperature scaling of Γ as T^3 is a behavior characteristic of a relativistic massless field theory in $2+1$ spacetime dimensions. This observed high-temperature behavior of the noncritical M-theory vacuum lends further support to the arguments of [1] that the collective excitations of the Fermi surface in this vacuum are effectively described by a bosonic collective field in $2+1$ dimensions.

At very high temperatures, the physics of the noncritical M-theory vacuum is governed by a dual effective theory. We claim that this dual theory is another solution of noncritical M-theory in $2+1$ dimensions, effectively at zero temperature, and with the effective M-theory coupling constant given by

$$\widetilde{g}_M = 1/\widetilde{\mu}. \tag{2.32}$$

This claim will be further substantiated by a duality argument in the next subsection. Here, we present first evidence for this claim. (2.30) implies that the partition function $\widetilde{\Gamma}(\widetilde{\mu})$ of the effective dual theory that governs the classical limit at high temperature should satisfy

$$\frac{\partial^3\widetilde{\Gamma}}{\partial\widetilde{\mu}^3} \sim -\frac{1}{2}\tanh(\pi\widetilde{\mu}). \tag{2.33}$$

This can be asymptotically expanded in the powers of $\widetilde{g}_M$,

$$\frac{\partial^3\widetilde{\Gamma}}{\partial\widetilde{\mu}^3} \sim -\frac{1}{2} + \text{nonperturbative terms}. \tag{2.34}$$

As a result, the leading term in the asymptotic expansion of $\widetilde{\Gamma}(\widetilde{\mu})$ in the powers of $1/\widetilde{\mu}$ is proportional to $\widetilde{\mu}^3$, and the asymptotic expansion is one-loop exact. As we saw in [1],

and again in Section 2.5 of this paper, this behavior is indeed a hallmark of noncritical M-theory in $2+1$ dimensions.

We can gain further insight into the nature of this dual solution of M-theory as follows. As in any other solution of the double-scaled Fermi liquid theory, the right-hand side of (2.33) should be equal to the derivative of the effective density of states $\widetilde{\rho}(\widetilde{\mu})$ of this solution. Thus, integrating (2.29) once, we obtain – up to a nonuniversal integration constant – the effective density of states of the dual solution that represents the high-temperature limit of the M-theory vacuum,

$$\widetilde{\rho}(\widetilde{\mu}) \sim -\frac{1}{2\pi}\log\cosh(\pi\widetilde{\mu}). \tag{2.35}$$

Thus, even though this density of states exhibits the leading asymptotic behavior characteristic of noncritical M-theory, it is strictly distinct from the density of states (2.4) of the original vacuum at zero temperature, demonstrating that noncritical M-theory is not self-dual under the exchange of the high and low temperature regions.

Since the effective dual theory at high temperatures is another solution of noncritical M-theory, our arguments from Section 2.5 imply that the effective loop-counting coupling in this solution is given by $\widetilde{g}_M^3$. This is related to the coupling g_M^3 of the original vacuum by

$$\widetilde{g}_M^3 = (2\pi)^3\, g_M^3\, T^3. \tag{2.36}$$

Thus, the effective loop-counting coupling in the high-temperature limit scales as T^3. This behavior seems to be a novel signature of M-theory, to be contrasted with the behavior in quantum field theory as well as string theory, as reviewed above. It would be very interesting to see whether a similar $\sim T^3$ scaling can also be found in full eleven-dimensional M-theory and its compactifications.

To summarize, the answers to the three questions about the high temperature behavior of noncritical M-theory are as follows:

(1) In the high temperature limit, the M-theory vacuum is effectively described by another solution of noncritical M-theory, effectively at zero temperature, and with the density of states given by (2.35).

(2) The free energy in the noncritical M-theory vacuum scales at high temperatures as T^3, *i.e.*, as in a massless field theory in $2+1$ dimensions.

(3) The scaling (2.36) of the effective coupling constant with T is unlike in field or string theory, suggesting that in this phase, noncritical M-theory is not (manifestly) equivalent to either.

2.8. Effective M-Theory at High Temperature from T-Duality

In the previous subsection, we found signs indicating that the high-temperature limit of the noncritical M-theory vacuum is effectively described by another solution of noncritical M-theory in $2+1$ dimensions. However, the exact nature of that solution was

left somewhat obscure. Now we will use T-duality of Type 0A and 0B string theories to identify this dual solution of noncritical M-theory.

As was reviewed in Section 2.1, the Hamiltonian of noncritical M-theory in the second-quantized fermionic representation can be expressed as an infinite sum of sectors with all possible integer values q of the angular momentum J on the plane. For each fixed q, the Hamiltonian is equivalent to that of the linear dilaton vacuum of Type 0A string theory with RR flux equal to q. In the vacuum state of noncritical M-theory, all available states are filled by fermions up to some common value μ of the Fermi energy in all sectors independently of the value of q. Using this decomposition, many physical quantities of the noncritical M-theory vacuum can be formally evaluated as infinite sums of contributions from Type 0A vacua with all possible values of the RR flux q. For example, the vacuum energy of the vacuum solution equals the sum of vacuum energies of Type 0A vacua of all integer values of q, all filled up to the common value of the Fermi energy μ. Similarly, the density of states (2.4) in M-theory can be evaluated as a sum over q of densities of states in Type 0A vacua at fixed q,

$$\rho(\xi)=\frac{1}{4\pi}\,\mathrm{Re}\int d\tau e^{-i\xi\tau}\,\frac{1}{\sinh^2(\tau/2)}=\sum_{q\in\mathbf{Z}}\frac{1}{2\pi}\,\mathrm{Re}\int d\tau e^{-i\xi\tau}\,\frac{1}{\sinh(\tau)}e^{-|q|\tau}. \tag{2.37}$$

Consider now noncritical M-theory at finite temperature T. Using (2.37), the thermodynamic potential Γ can be written as an infinite sum of contributions from sectors of fixed angular momentum q. Recalling the integral formula (2.6), we can write

$$\begin{aligned}2\pi R\cdot\Gamma&==-\frac{1}{4}\,\mathrm{Re}\int_0^\infty\frac{d\tau}{\tau}\,e^{i\mu\tau}\frac{1}{\sinh^2(\tau/2)}\,\frac{1}{\sinh\left[\tau/(2R)\right]}\\&=-\frac{1}{2}\sum_{q\in\mathbf{Z}}\,\mathrm{Re}\int_0^\infty\frac{d\tau}{\tau}\,e^{i\mu\tau}\frac{1}{\sinh(\tau)}\,\frac{1}{\sinh\left[\tau/(2R)\right]}\,e^{-|q|\tau}.\end{aligned} \tag{2.38}$$

Each contribution of fixed q in (2.38) is equivalent to $2\pi R\cdot\Gamma$ evaluated in Type 0A theory with RR flux q, at temperature T.

In the canonical ensemble, Type 0A theory at temperature T is described by the solution with the Euclidean time compactified on a circle of radius $R=1/(2\pi T)$. It was shown in [27] that this Euclidean solution of Type 0A string theory is T-dual to a Type 0B solution at the dual values of the radius and of the inverse string coupling,[7]

$$\widetilde{R}=\frac{1}{2R},\qquad\widetilde{\mu}=R\,\mu. \tag{2.39}$$

[7] For some earlier work on T-duality in two-dimensional Type 0A and 0B string vacua, see [34-36].

In addition, this dual Type 0B solution exhibits a nonzero value ν of the one-form RR flux, given by [27]

$$\nu = \frac{iq}{2\widetilde{R}}. \tag{2.40}$$

At finite $\widetilde{R}$, the unit of the Type 0B RR flux is quantized, and the flux becomes continuous in the decompactification limit $\widetilde{R} \to \infty$.

In our expression (2.38) for the thermodynamic potential, we can now simultaneously perform T-duality on all the Type 0A string theories parametrized by the value of q. $\Gamma(R, \mu)$ is thus mapped to

$$\begin{aligned} 2\pi R \cdot \Gamma &= -\frac{1}{2} \sum_{q \in \mathbf{Z}} \mathrm{Re} \int \frac{d\sigma}{\sigma}\, e^{i\tilde{\mu}\sigma} \frac{1}{\sinh(\sigma/2)} \frac{1}{\sinh[\sigma/(2\widetilde{R})]} e^{-|q|\sigma/(2\tilde{R})} \\ &= -\frac{1}{4} \mathrm{Re} \int \frac{d\sigma}{\sigma}\, e^{i\tilde{\mu}\sigma} \frac{1}{\sinh(\sigma/2)} \frac{1}{\sinh^2[\sigma/(4\widetilde{R})]}. \end{aligned} \tag{2.41}$$

In this way, the thermodynamic potential has been rewritten as an infinite sum over q of contributions from *Type 0B vacua* with RR flux equal to $\nu = iq/(2\widetilde{R})$. Indeed, with the help of [9] and [27], the individual terms in the sum in (2.41) can indeed be recognized as the partition functions of the corresponding Type 0B vacua with flux ν.

As we pointed out in Section 2.1, the Hamiltonian of noncritical M-theory can also be written as an integral over a one-parameter family of Type 0B theories parametrized by the continuous eigenvalue ν of the single-particle Hamiltonian h_2 of (say) the second of the two one-dimensional oscillators. When we put the theory on a Euclidean time circle of radius $\widetilde{R}$, the eigenvalues ν become purely imaginary and quantized in the units of $1/\widetilde{R}$, and the integral over ν turns into a discrete sum. This suggests that the eigenvalue of h_2 in the Cartesian representation of the Fermi liquid describing noncritical M-theory should be identified with the Type 0B RR flux, much like the Type 0A RR flux q was identified with the eigenvalue of the angular momentum J in the polar-coordinate representation of the theory. It also implies that noncritical M-theory inherits a T-duality symmmetry from the underlying T-duality between Type 0A and 0B strings. In noncritical M-theory, this T-duality swaps the Cartesian and polar-coordinate representations, in which the theory is decomposed into a collection of Type 0A or Type 0B string theories will all possible values of the corresponding RR flux.

In the infinite temperature limit, $\tilde{R}$ goes to infinity, the Type 0B flux ν becomes continuous, and the sum in (2.41) turns into an integral. In this limit, the density of states of the dual M-theory solution can be evaluated, and we obtain

$$\widetilde{\rho}(\widetilde{\mu}) \equiv \frac{\partial^2 \widetilde{\Gamma}}{\partial \widetilde{\mu}^2} \sim -\frac{1}{2\pi} \log\cosh(\pi\widetilde{\mu}) + \frac{\Lambda}{2}. \tag{2.42}$$

This reproduces our prediction (2.35) for the density of states of the effective high-temperature theory.

Thus, our conclusions are as follows:

- Noncritical M-theory inherits a T-duality symmetry from the underlying T-duality between the Type 0A and 0B strings.
- The high-temperature regime of the M-theory vacuum is described by the T-dual solution of noncritical M-theory, at the dual value of the temperature.

Having established that the high-temperature regime is related to the low-temperature regime by T-duality, we can now return to the original task of Section 2.7 and study the high-temperature expansion of the thermodynamic potential Γ of the original M-theory vacuum. This is an expansion in the powers of R with $\widetilde{\mu}$ fixed. From the perspective of the T-dual theory, this is a low-temperature expansion in the powers of $1/\widetilde{R}$ with $\widetilde{\mu}$ fixed, analogous to the low-temperature expansion of the original vacuum discussed in Section 2.6. We obtain

$$\begin{aligned}
\frac{\partial^3\Gamma}{\partial\widetilde{\mu}^3} &= -\frac{1}{8\pi R}\int_0^\infty d\sigma\,\sigma^2\,\sin(\widetilde{\mu}\sigma)\,\frac{1}{\sinh^2(\sigma R/2)\,\sinh(\sigma/2)}\\
&\approx \frac{1}{2\pi R^3}\sum_{k=0}^\infty \frac{(2k-1)B_{2k}R^{2k}}{(2k)!}\int_0^\infty d\sigma\,\sin(\widetilde{\mu}\sigma)\,\frac{\sigma^{2k}}{\sinh(\sigma/2)}\\
&\approx \frac{1}{2\pi R^3}\sum_{k=0}^\infty \frac{(2k-1)B_{2k}R^{2k}}{(2k)!}(-1)^k\,\frac{\partial^{2k}}{\partial\widetilde{\mu}^{2k}}\int_0^\infty d\sigma\,\sin(\widetilde{\mu}\sigma)\,\frac{1}{\sinh(\sigma/2)}\\
&\approx \frac{1}{2R}\sum_{k=0}^\infty \frac{(2k-1)B_{2k}R^{2k-2}}{(2k)!}(-1)^k\,\frac{\partial^{2k}}{\partial\widetilde{\mu}^{2k}}\,\tanh(\pi\widetilde{\mu}).
\end{aligned}\tag{2.43}$$

This can be summarized in a compact formula analogous to our low-temperature expansion formula (2.24),

$$\frac{\partial^3\Gamma}{\partial\widetilde{\mu}^3} \approx -\frac{\left(\frac{\partial}{\partial\widetilde{\mu}}\right)^2}{8R\,\sin^2\left(\frac{R}{2}\frac{\partial}{\partial\widetilde{\mu}}\right)}\,\tanh(\pi\widetilde{\mu}).\tag{2.44}$$

The astute reader may recognize in (2.43) the strong resemblance to the partition function of the A-model topological closed string theory on the resolved conifold (see, *e.g.*, [37,38] for reviews), with R essentially playing the role of the string coupling of the A-model, and $\widetilde{\mu}$ related to the Kähler modulus of the conifold. This is our first hint of a much deeper relation between noncritical M-theory compactified on a Euclidean time circle, and the strings of the topological A-model. This relation will be the topic of Section 3 below.

2.9. Noncritical M-Theory at the Debye Temperature

It was noticed in [1] that the exact vacuum energy $E_0(\mu)$ of the vacuum solution in noncritical M-theory at zero temperature is mathematically equivalent to the partition

function of Debye phonons in a Debye crystal at temperature $T_D = 1/(2\pi)$,

$$E_0(\mu) = \frac{1}{2}\int_\mu^\Lambda d\xi\,\xi\,\rho_D(\xi)\left(\frac{1}{e^{2\pi\mu}-1}+\frac{1}{2}\right), \tag{2.45}$$

with the Debye density of states $\rho_D(\xi) \sim \xi$ characteristic of a $2+1$ dimensional system. In a slight detour from the main theme of this paper, we shall now study the partition function of the noncritical M-theory vacuum at this "Debye temperature."

The Debye temperature corresponds to the compactification radius $R = 1$ of the Euclidean time dimension. In noncritical string theory, this particular value of R plays a special role, since the theory at this radius is related to the Penner model [39].

At $R = 1$, the first derivative of the thermodynamic potential (2.3) of our noncritical M-theory vacuum can be evaluated by residues, leading to

$$\frac{\partial\Gamma(\mu)}{\partial\mu} = -\frac{1}{4}\left(\mu^2+\frac{1}{4}\right)\tanh(\pi\mu)+\frac{\Lambda\mu}{2}+\frac{\Lambda^2}{2}. \tag{2.46}$$

This can be further integrated, yielding the following exact expression for $\Gamma(\mu)$,

$$\begin{aligned}\Gamma(\mu) = &-\frac{1}{12}\,\mu^3+\frac{\Lambda}{4}\,\mu^2+\left(\frac{\Lambda^2}{2}-\frac{1}{16}\right)\,\mu+C(\Lambda)\\ &+\frac{1}{4}\sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{k\pi}\,\mu^2+\frac{1}{k^2\pi^2}\,\mu+\frac{1}{4k\pi}+\frac{1}{2k^3\pi^3}\right)e^{-2k\pi\mu}.\end{aligned} \tag{2.47}$$

This formula nicely illustrates all of the features of the asymptotic expansion in the powers of g_M that were discussed in Section 2.5 above. In particular, $\Gamma(\mu)$ contains a perturbative series which is one loop exact and starts off with the classical contribution at order $\mu^3 = 1/g_M^3$. This series is followed by an infinite series of instanton-like contributions. In each of the instanton terms, the perturbative quantum corrections are also one-loop exact, with the leading classical term vanishing, and the lowest nontrivial quantum correction starting at order $\mu^2 = 1/g_M^2$.

Using (2.46), and dropping the nonuniversal terms, $\Gamma(\mu)$ can also be rewritten in another interesting form, as

$$\Gamma(\mu) = \frac{1}{2}\int d\mu\,(\mu^2+\frac{1}{4})\left(\frac{1}{e^{2\pi\mu}+1}-\frac{1}{2}\right). \tag{2.48}$$

In this form, the universal part of the thermodynamic potential of noncritical M-theory at the Debye temperature is mathematically equivalent to the energy of a system of *fermions* in $2+1$ dimensions at the Debye temperature, and with the effective density of states $\rho_D(\mu) \sim 2\mu + 1/2\mu$. It is intriguing that this effective density of states is self-dual under an inversion of μ.

3. Duality to the Closed String Theory of the Topological A-Model

Our analysis of the partition function of noncritical M-theory at finite temperature in Section 2 has revealed a surprising connection with the amplitudes of the topological closed string of the A-model on the resolved conifold. In this section, we will attempt to make this duality more precise, and analyze some of its possible implications.

3.1. First Comparison of the Partition Functions

In Section 2.8, we derived the asymptotic high-temperature expansion of the partition function $\mathcal{Z}_M$ of the noncritical M-theory vacuum,

$$\frac{\partial^3 \log \mathcal{Z}_M}{\partial \widetilde{\mu}^3} \approx -\pi \sum_{k=0}^{\infty} \frac{(2k-1)B_{2k}(iR)^{2k-2}}{(2k)!} \frac{\partial^{2k}}{\partial \widetilde{\mu}^{2k}} \left(\tanh(\pi\widetilde{\mu})\right). \tag{3.1}$$

This can be rewritten in terms of polylogarithms as

$$\frac{\partial^3 \log \mathcal{Z}_M}{\partial \widetilde{\mu}^3} \approx \frac{2\pi}{(iR)^2}\left(\frac{1}{2} + \mathrm{Li}_0(-e^{-2\pi\tilde{\mu}})\right) - \sum_{k=1}^{\infty}(2\pi i R)^{2k-2}\frac{(2\pi)^3 B_{2k}}{2k(2k-2)!}\,\mathrm{Li}_{-2k}(-e^{-2\pi\tilde{\mu}}). \tag{3.2}$$

We now wish to integrate this expression to get an expansion of $\log \mathcal{Z}_M$. (3.2) determines $\log \mathcal{Z}_M$ up to a second-order polynomial in $\widetilde{\mu}$, whose coefficients could be functions of R containing non-universal dependence on the cutoff Λ. The universal dependence of this polynomial on R can be determined by taking derivatives of $\log \mathcal{Z}_M$ with respect to β, in exact parallel with our analysis of the R dependence of the polynomial (2.7) in Section 2.3. In the end, the nonuniversal cutoff dependence afflicts only a few lowest-order terms in the double expansion in R and $\widetilde{\mu}$, and we obtain

$$\begin{aligned}\log \mathcal{Z}_M \approx \frac{1}{(2\pi i R)^2} &\left(p(\widetilde{\mu},\Lambda) + \frac{(2\pi\widetilde{\mu})^3}{12} - \mathrm{Li}_3(-e^{-2\pi\tilde{\mu}})\right)\\ &+\left(C(\Lambda) - \frac{\pi\widetilde{\mu}}{12} - \frac{1}{12}\log(1+e^{-2\pi\tilde{\mu}})\right)\\ &+\sum_{k=2}^{\infty}(2\pi i R)^{2k-2}\frac{B_{2k}}{2k(2k-2)!}\,\mathrm{Li}_{3-2k}(-e^{-2\pi\tilde{\mu}}).\end{aligned} \tag{3.3}$$

Here $p(\widetilde{\mu},\Lambda)$ is a nonuniversal polynomial of second order in $\widetilde{\mu}$, and $C(\Lambda)$ is a nonuniversal constant; both $p(\widetilde{\mu},\Lambda)$ and $C(\Lambda)$ are independent of R.

In this form, the close similarity to the partition function of the closed topological string of the A-model on the resolved conifold is apparent. Indeed, recall (or see in [28,29,37,38]) that the partition function of the A-model on the resolved conifold can be

written as an asymptotic expansion in the powers of the string coupling constant g_A, with coefficients being functions of the Kähler modulus t_A,

$$\begin{aligned}\log \mathcal{Z}_A \approx \frac{1}{g_A^2}\left(p_A(t_A)+\frac{t_A^3}{12}-\operatorname{Li}_3(e^{-t_A})\right)+\left(C_A-\frac{t_A}{24}-\frac{1}{12}\log(1-e^{-t_A})\right)\\ +\sum_{h=2}^{\infty} g_A^{2h-2}\left(\frac{B_{2k}B_{2k-2}}{2k(2k-2)(2k-2)!}+\frac{B_{2k}}{2k(2k-2)!}\operatorname{Li}_{3-2k}(e^{-t_A})\right).\end{aligned} \tag{3.4}$$

Here $p_A(t_A)$ is a non-universal second-order polynomial in t_A, and C_A is a non-universal constant. The real part of the Kähler modulus t_A measures the size of the S^2 of the resolved conifold, and its imaginary part is given by the B-field flux through this S^2. The t_A-independent terms in (3.4) come from constant maps from the string worldsheet to the target conifold, and the polylogarithms come from worldsheet instantons.

A first comparison of (3.3) and (3.4) suggests the identification

$$g_A = 2\pi i R, \qquad t_A = 2\pi\widetilde{\mu} + \pi i. \tag{3.5}$$

The matching between $\mathcal{Z}_M$ and $\mathcal{Z}_A$ is close, but there is an apparent difference: The constant term in (3.4) at each order in the A-model string coupling is absent in the partition function of the vacuum of noncritical M-theory.

We claim that this difference is purely due to the fact that the two partition functions $\mathcal{Z}_A$ and $\mathcal{Z}_M$ are normalized differently. Indeed, notice that the partition function of the A-model comes out naturally normalized so that the genus-h term in $\mathcal{Z}_A$ diverges as t_A^{2-2h}, all subleading terms going to zero as $t_A \to 0$. This is in accord with the Gopakumar-Vafa duality [28,29], which maps the A-model to $U(N_c)$ Chern-Simons gauge theory on S^3. In order to check this behavior, we can expand the polylogarithms around $t_A = 0$, using the asymptotic expansion formula

$$\begin{aligned}\operatorname{Li}_{3-2k}(e^{-t_A}) &\approx \Gamma(2k-2)t_A^{2-2k}+\sum_{n=0}^{\infty}\frac{(-1)^n\zeta(3-2k-n)}{n!}t_A^n\\ &\approx (2k-3)!\,t_A^{2-2k}-\frac{B_{2k-2}}{2k-2}+\mathcal{O}(t_A), \qquad k=2,3,\ldots.\end{aligned} \tag{3.6}$$

Thus, the polylogarithm terms in (3.4), which come from the sum over worldsheet instantons of genus k, have a leading divergence $\sim t_A^{2-2k}$, followed by a subleading constant term as t_A goes to zero. From the point of view of the Gopakumar-Vafa duality, the leading divergence is the nonperturbative term (discussed for example in [40]) which is not captured in the 't Hooft expansion of the dual Chern-Simons theory. At each order in g_A, the term originating from the constant maps in (3.4) is precisely such that it subtracts the subleading constant from the instanton polylogarithms, ensuring that – order by order in g_A – $\log \mathcal{Z}_A$ vanishes at $t_A = 0$ after the leading $\sim t_A^{2-2h}$ divergence has been subtracted.

On the other hand, the partition function of noncritical M-theory is naturally normalized so that it simplifies in a different limit, of $\widetilde{\mu} \to \infty$. If it were not for the divergence in (2.3), setting $\mu = \infty$ in this expression would lead to $\mathcal{Z}_M = 1$. This is again intuitively clear, since sending μ to infinity corresponds to emptying the Fermi sea; in the absence of any fermions, the partition function should be equal to one. The divergence of $\log \mathcal{Z}_M$ makes the precise realization of this formal expectation slightly subtle: One must empty the Fermi sea simultaneously with the double-scaling limit (*i.e.*, to set $\mu \sim \Lambda$ as $\Lambda \to \infty$) in order to ensure that $\mathcal{Z}_M = 1$ as $\widetilde{\mu} \to \infty$.

When we normalize $\mathcal{Z}_M$ and $\mathcal{Z}_A$ so that they are equal to one for the same (but otherwise arbitrarily chosen) fixed value $t_{A,\mathrm{fix}}$ of t_A, they turn out to be equal as asymptotic series in t_A. In this sense, $\mathcal{Z}_A$ and $\mathcal{Z}_M$ carry precisely the same information, and we can summarize our result in the following duality relation between the partition functions:

$$\frac{\mathcal{Z}_A(g_A, t_A)}{\mathcal{Z}_A(g_A, t_{A,\mathrm{fix}})} = \frac{\mathcal{Z}_M(R, \widetilde{\mu})}{\mathcal{Z}_M(R, \widetilde{\mu}_{\mathrm{fix}})}, \tag{3.7}$$

with the parameters on the two sides related by (3.5) (in particular, $t_{A,\mathrm{fix}} = 2\pi\widetilde{\mu}_{\mathrm{fix}} + \pi i$).

It is worth stressing again that since the A-model is only known as an asymptotic series in g_A, the duality relation (3.7) is to be only interpreted as an equality between two asymptotic expansions. However, within the framework of noncritical M-theory, $\mathcal{Z}_M$ is defined nonperturbatively in R, and therefore represents a possible nonperturbative completion of the topological A-model.

3.2. A More General Form of the Duality

In the thermodynamic ensemble, the partition function corresponds to fermions antiperiodic around the Euclidean time dimension t_E. Since the thermodynamic interpretation is no longer important in this section, we shall relax this condition, and allow for an arbitrary phase b in the periodicity of the fermions,

$$\Psi(t_E + 2\pi R, \lambda_1, \lambda_2) = e^{ib}\Psi(t_E, \lambda_1, \lambda_2). \tag{3.8}$$

The case of the thermal ensemble studied in Section 2 corresponds to $b = \pi$.

How does the general phase appear in the partition function? In the first-quantized framework that we used to calculate $\log \mathcal{Z}$ in (2.3), allowing for the general phase in (3.8) amounts to the insertion of an extra factor of $(-e^{ib})^F$, with F the fermion number. (The minus sign is due to the fact that (2.3) calculates $\log \mathcal{Z}$ for the thermal case of $b = \pi$.) Hence, the partition function with arbitrary periodicity of the fermions becomes

$$\log \mathcal{Z}_M(R, \widetilde{\mu}, b) = \int_{-\infty}^{\infty} d\xi\, \rho(\xi) \log\left(1 - e^{ib} e^{2\pi\tilde{\mu} - 2\pi R\,\xi}\right). \tag{3.9}$$

Expanding this in the asymptotic series in R, we obtain

$$\begin{aligned}\log \mathcal{Z}_M \approx \frac{1}{(2\pi i R)^2}\left(p(\widetilde{\mu},b,\Lambda)+\frac{(2\pi\widetilde{\mu}+ib)^3}{12}-\,\mathrm{Li}_3(e^{-2\pi\tilde{\mu}-ib})\right)\\ +\left(C(b,\Lambda)+\frac{2\pi\widetilde{\mu}+ib}{24}+\frac{1}{12}\log(1-e^{-2\pi\tilde{\mu}-ib})\right)\\ -\sum_{k=2}^{\infty}(2\pi i R)^{2k-2}\frac{B_{2k}}{2k(2k-2)!}\,\mathrm{Li}_{3-2k}(e^{-2\pi\tilde{\mu}-ib}).\end{aligned} \tag{3.10}$$

This again matches $\log \mathcal{Z}_A$ modulo normalization, but now the entire range of complex values of the Kähler modulus t_A is allowed, with the phase b playing the role of the imaginary part of t_A.

Recalling that in the original variables $\widetilde{\mu}$ is equal to $R\mu$, and that $g_M = 1/\mu$ plays the role of the coupling constant in the noncritical M-theory vacuum, we find the duality relations

$$g_A = 2\pi i R, \qquad t_A = 2\pi\widetilde{\mu}+ib = \frac{2\pi R}{g_M}+ib \tag{3.11}$$

advertized in the introduction. With this matching of the parameters, the partition function are related by a generalization of (3.7) to the case of arbitrary b,

$$\frac{\mathcal{Z}_A(g_A,t_A)}{\mathcal{Z}_A(g_A,t_{A,\mathrm{fix}})} = \frac{\mathcal{Z}_M(R,\widetilde{\mu},b)}{\mathcal{Z}_M(R,\widetilde{\mu}_{\mathrm{fix}},b_{\mathrm{fix}})}. \tag{3.12}$$

We have again chosen an arbitrary normalization point $t_{A,\mathrm{fix}}$, with $\widetilde{\mu}_{\mathrm{fix}}$ and b_{fix} related to $t_{A,\mathrm{fix}}$ via (3.11).

3.3. Solutions with the Universal Bottom of the Fermi Sea

As we have just seen, the partition functions of the vacuum of noncritical M-theory and of the topological A-model on the resolved conifold are virtually identical, differing only in their normalization properties. Here we wish to point out the existence of a larger family of solution of noncritical M-theory, whose partition functions are naturally normalized in a way similar to that of the partition function of the A-model.

The conventional vacuum of noncritical M-theory corresponds to a large N limit of the system of N fermions occupying all single-particle states from the cutoff $-\Lambda$ up to the top of the Fermi sea at the double-scaled Fermi energy μ. Recall that in Section 7.4 of [1], we studied a broader class of noncritical M-theory solutions, characterized by two Fermi surfaces: The universal top of the sea at some $\mu = \mu_+$, and the universal bottom of the sea at some $\mu_- > \mu_+$ (see Figure 3). This construction gives a two-parameter family of static solutions of noncritical M-theory, parametrized by $\mu_\pm$.

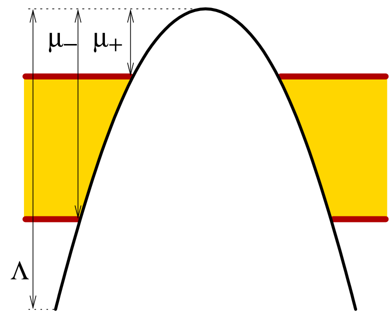

Fig. 3: The solution of noncritical M-theory with a universal bottom of the Fermi sea. In this solution, all single-particle states are occupied between the top top of the Fermi sea at μ_+ and the bottom of the sea at μ_-. (As in Fig. 1, the angular dimension of the spatial plane has been suppressed.)

Consider now this family of solution in Euclidean signature, with the Euclidean time compactified on a circle of radius R, and define $\widetilde{\mu}_\pm = 2\pi R\,\mu_\pm$. The special case of particular interest to us is

$$\widetilde{\mu}_+ = -\widetilde{\mu}, \qquad \widetilde{\mu}_- = 0. \tag{3.13}$$

Thus, the Fermi sea is filled from the top of the potential at zero, to some Fermi energy μ *above* the top of the potential.

The partition function $\hat{\mathcal{Z}}_M(R,\widetilde{\mu})$ of this solution of noncritical M-theory no longer approaches one at $\widetilde{\mu}\to\infty$. Instead, it simplifies in the limit of $\widetilde{\mu}\to 0$, which corresponds to emptying the Fermi sea. It would be interesting to study this extended class of solutions of noncritical M-theory for general values of $\widetilde{\mu}_\pm$, and identify their relation to the topological A-model. This is, however, beyond the scope of this paper.

4. Consequences of the Duality

The proposed duality between noncritical M-theory and closed string theory of the topological A-model exhibits several notable features, and can be expected to have interesting implications for a wide range of phenomena traditionally related to the topological A-model. Here we present some comments and raise some questions about the possible implications of this duality.

As we saw in Section 3.2, the parameters on the two sides of the duality are related as follows,

$$g_A = 2\pi i R, \qquad t_A = \frac{2\pi R}{g_M} + ib. \tag{4.1}$$

In particular, the duality relates the coupling constant g_M of noncritical M-theory to the Kähler modulus of the resolved conifold on the dual side. Thus, the dimensionless

coupling constant of noncritical M-theory acquires a geometric interpretation, as promised in Section 2.5.

From the point of view of the dual A-model, the M-theory coupling constant g_M plays the role of a worldsheet coupling. In [1] (and again in Section 2.5), we found the characteristic leading behavior $\sim 1/g_M^3$ of the free energy of the noncritical M-theory vacuum at weak coupling. Our duality (4.1) provides an intriguing explanation for this leading power of $1/g_M^3 \sim N^3$ in the large-N behavior of noncritical M-theory, by relating it to the triple intersection form of the Calabi-Yau, which gives a tree-level contribution in the dual A-model [37,38].

Topological M-theory for the A-model:

The duality (4.1) relates the string coupling constant g_A of the A-model to a geometric parameter – the radius of the Euclidean time dimension – on the dual side of noncritical M-theory. Hence, the A-model string coupling g_A acquires a geometric interpretation, as the size of the "M-theory circle." Such a relation $g_A \sim R$ between the string coupling and the radius of the M-theory circle is one of the hallmarks of the string theory/M-theory duality in critical string theory. This suggests that noncritical M-theory could be a realization of the idea of "topological M-theory," as proposed for closed topological string theory of the A-model on Calabi-Yau in [25] (see also [26]).

In this context, it is intriguing that the role of the M-theory dimension is played by the Euclidean *time* dimension of noncritical M-theory. One motivation for introducing topological M-theory was to explain the apparent behavior of the partition function of the A-model as a wave function of a system evolving in an extra time dimension. Perhaps, the factor of i in the relation (4.1) between the string coupling g_A and the radius R is an indication that the duality should be more naturally interpreted after the Wick rotation to real time, effectively replacing R with iR and reinterpreting R as the time lapse in real time evolution.

Relation to Chern-Simons theory on S^3:

Gopakumar-Vafa duality [28,29] relates the A-model on the resolved conifold to the $U(N_c)$ Chern-Simons gauge theory on S^3. The Gopakumar-Vafa relations are

$$g_A = \frac{2\pi i}{k + N_c}, \qquad t_A = \frac{2\pi i N_c}{k + N_c}, \tag{4.2}$$

where k is the level of the Chern-Simons theory.

When combined with the Gopakumar-Vafa duality, our duality implies a relation between noncritical M-theory and $U(N_c)$ Chern-Simons gauge theory on S^3. For the quantities in noncritical M-theory, this implies

$$R = \frac{1}{k + N_c}, \qquad \mu + \frac{ib}{2\pi R} = iN_c. \tag{4.3}$$

These relations have a very natural interpretation, with R being the parameter of the large-N_c expansion, *i.e.*, the gauge coupling constant g_{CS}^2. Moreover, if we choose to perform the Wick rotation to real time, (4.3) implies an intriguing relation $g_M = 1/N_c$, identifying the M-theory coupling constant with the inverse number of colors in the dual Chern-Simons gauge theory. Note also that since the Chern-Simons theory can be reinterpreted as a matrix model [41], this connection leads to a matrix model description of noncritical M-theory.

Nonperturbative completion of the A-model:

Since the partition function of the topological strings of the A-model is defined only as an asymptotic expansion in g_A, relations between partition functions implied by (3.7) are understood as equalities between asymptotic expansions. However, the partition function $\mathcal{Z}_M$ of noncritical M-theory is nonperturbatively defined, and therefore (3.7) can be interpreted as a nonpertutbative completion of the asymptotic series for the A-model.

Another nonperturbative completion of the A-model has emerged recently in the context of the OSV conjecture [42]. This conjecture relates the entropy of certain supersymmetric black holes to the absolute value squared of a topological string partition function, and its precise form is still being developed. It should be interesting to investigate the relation between the nonperturbative completions of the A-model via the OSV conjecture and via the duality to noncritical M-theory. In particular, the OSV conjecture has been tested in [43,44] for the topological string on a noncompact Calabi-Yau given by the $\mathcal{O}(-p) \times \mathcal{O}(p+2g-2)$ fibration over a genus-g Riemann surface Σ_g. The partition functions are related to the large N_c, q-deformed version of Yang-Mills on Σ_g. In turn, two-dimensional large-N_c Yang-Mills theory on Σ_g is known to have an interpretation in terms of a string theory [45], whose worldsheet description is given by the topological rigid string [46] (see also [47]). The case of $p = 1$ and $g = 0$ is rather singular, and somewhat outside of the scope of [44]. In particular, the large-N_c Yang-Mills theory on S^2 is known to exhibit the Douglas-Kazakov phase transition [48].[8]

However, $p = 1$ and $g = 0$ is the resolved conifold, for which the duality to noncritical M-theory represents a nonperturbative completion of the A-model, and could therefore be complementary to the methods of [44]. Note in particular that the partition function (3.9) of noncritical M-theory exhibits a singular behavior at $b = 0$, with each term of the asymptotic expansion (3.10) divergent at $2\pi\widetilde{\mu} + ib = 0$. We expect this singular behavior to be related to the Douglas-Kazakov phase transition of Yang-Mills theory on S^2.

S-duality of the A-model from T-duality of noncritical M-theory:

As we saw in Section 2, noncritical M-theory can be formally decomposed into an infinite number of sectors equivalent to Type 0A or 0B theory with all possible values of

[8] In the context of the OSV conjecture, this phase transition has been recently studied in [49].

their RR flux. When compactified on the thermal circle, noncritical M-theory inherits a duality symmetry from the T-duality between the Type 0A and 0B string theories. This duality was used in Section 2.8 to shed light on the high-temperature behavior of noncritical M-theory. It acts on the parameters of the vacuum by

$$\widetilde{\mu} = \mu\, R, \qquad \widetilde{R} = \frac{1}{2R}. \tag{4.4}$$

When combined with the duality (4.1) to the topological A-model, this T-duality predicts the existence of an S-duality for the A-model. Indeed, in terms of the A-model variables, (4.4) acts by inverting the string coupling constant and rescaling the Kähler modulus,

$$g_A \to \frac{1}{g_A}, \qquad t_A \to t_A g_A, \tag{4.5}$$

i.e., as an S-duality transformation.

An S-duality for the A-model on a Calabi-Yau has been proposed in the literature [50,51]. It also acts by (4.5), and relates the A-model to the B-model on the same target and with its string coupling given by $g_B \sim 1/g_A$. It should be interesting to investigate the relationship between the S-duality suggested by noncritical M-theory and the S-duality proposed in [50,51]. The latter has been related to T-duality in critical string theory in [52].

The melting crystal and quantum foam:

At infinite t_A, string theory of the topological A-model has been related to a classical crystal melting problem [30]. The inverse temperature of that crystal is proportional to g_A. In combination with our duality, (4.1) shows that the temperature of the melting crystal is essentially the temperature of the thermal ensemble in the noncritical M-theory vacuum. Moreover, in [1], we evaluated the exact vacuum energy of the noncritical M-theory vacuum, and found a surprising similarity with the partition function of phonons in a Debye crystal, with μ related to the size of this crystal. (4.1) relates μ to the Kähler modulus of the conifold, which was set to infinity in [30], corresponding to an infinite crystal in [30]. The case of finite t_A was studied in [53], with μ indeed providing a scale related to the size of the crystal. Given this matching of the physical interpretation of the parameters, it is natural to suspect that the constituents of the spacetime foam crystal of [30] are intimately related to the constituent fermions of noncritical M-theory.

Embedding into M-theory:

The relation between noncritical M-theory and the topological A-model was found in Section 3 by comparing the partition functions of the two theories. It would clearly be desirable to have a more systematic derivation of such a duality, by embedding noncritical M-theory to superstring theory or eleven-dimensional M-theory.

In the related case of the Gopakumar-Vafa duality, such a derivation exists [54]. It takes advantage of a relation between topological strings on Calabi-Yau and M-theory on a noncompact G_2 holonomy manifold. From the eleven-dimensional vantage point of M-theory, the duality is a simple consequence of the flop transition. In addition, G_2 holonomy manifolds are also central ingredients in the conjectured relation of topological strings on Calabi-Yau to topological M-theory in seven dimensions.

We expect M-theory on G_2 holonomy manifolds to be relevant for the duality between noncritical M-theory and the topological A-model as well. Indeed, in the case of the Gopakumar-Vafa duality [54], the G_2 holonomy 7-manifold in question is given by a quadratic equation in $\mathbf{C}^4$,

$$|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = 2\mu. \tag{4.6}$$

This should be compared to the Fermi surface of the vacuum state in noncritical M-theory. In the double scaling limit, the Fermi liquid describing noncritical M-theory in $2+1$ dimensions becomes semiclassical. The semiclassical Fermi surface is a hypersurface in the classical phase space parameterized by coordinates λ_1, λ_2 and momenta p_1, p_2. In particular, the vacuum state of noncritical M-theory (at zero temperature) is described by the Fermi surface satisfying[9]

$$p_1^2 + p_2^2 - \lambda_1^2 - \lambda_2^2 = 2\mu. \tag{4.7}$$

Thus, the Fermi surface of the noncritical M-theory vacuum corresponds to the intersection of the G_2 holonomy 7-manifold (4.6) with the real section $\mathbf{R}^4 \subset \mathbf{C}^4$! This is very similar to the embedding of two-dimensional noncritical string theories into full string theory [55,56]. Interestingly, the flop transition of the G_2 holonomy manifold is realized by $\mu \to -\mu$, which in (4.7) corresponds precisely to the particle-hole duality in noncritical M-theory.

5. Conclusions

Noncritical M-theory in $2+1$ dimensions represents a unifying framework, in which various string theories are related by dualities, in a controlled setting of an exactly solvable model. In this paper, we have seen further evidence supporting this picture. In particular, we have found that noncritical M-theory is related to string theories in at least two different ways, via two distinct forms of M-theory/string theory duality.

When compactified on the Euclidean time circle, noncritical M-theory in $2+1$ dimensions has two $U(1)$ symmetries: the rotations of the spatial eigenvalue plane, and the

[9] For other solutions of the classical equations of motion of the Fermi surface in noncritical M-theory, see [1].

translations of the Euclidean time circle. The reduction on the angular S^1 on the eigenvalue plane is related to noncritical $\hat{c} = 1$ Type 0A theory. In this correspondence, the Kaluza-Klein charge of the reduction (*i.e.*, the angular momentum on the eigenvalue plane) is identified with the RR charge of stable D0-branes in Type 0A string theory. This was the original picture that led to the definition of noncritical M-theory in terms of a Fermi liquid in [1].

In Section 3 of this paper, we have seen evidence that the reduction of noncritical M-theory along the Euclidean time circle is also related to a string theory, namely the closed string theory of the topological A-model on the resolved conifold. In this correspondence, the radius R of the Euclidean time S^1 is interpreted as the string coupling g_A of the A-model.

Among the most intriguing features of noncritical M-theory (and of its cousins, matrix models of noncritical string theories) is the fact that the physical spacetime emerges as a derived concept, associated to the existence and geometrical properties of the Fermi surface. In this emergent picture of spacetime, the elementary fermions of the Fermi liquid – originating from D0-branes in the underlying string theory – are the fundamental constituents, and the smooth macroscopic effective geometry of spacetime is a collective phenomenon. The rigid eigenvalue plane populated by the fermions is an auxiliary structure, only rather indirectly related to the physical space. Thus, the exactly solvable setting of noncritical M-theory seems particularly suitable for extracting more lessons about the emergence and microscopic constituents of spacetime in string and M-theory.

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